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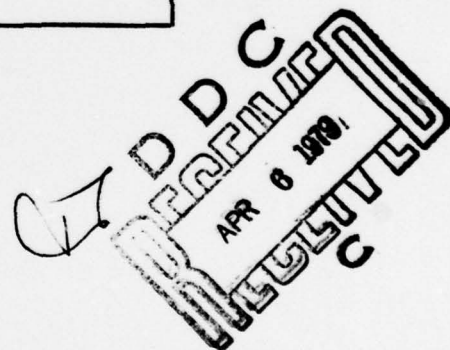## GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION
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⑨ Management Sciences Research Report No. 426

⑭ MSRR-426, WP-39-78-79

⑥ INTEGER AND FRACTIONAL MATCHINGS.

by

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⑪ Feb 79

⑫ 23p.

⑮ N00014-75-c-0621,
NSF-MPS73-08534

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Abstract

We examine the connections between maximum cardinality edge matchings in a graph and optimal solutions to the associated linear program, which we call maximum f-matchings (fractional matchings). We say that a maximum matching M separates an odd cycle with vertex set S, if M has no edge with exactly one end in S. An odd cycle is separable if it is separated by at least one maximum matching. We show that (1) a graph G has a maximum f-matching that is integer, if and only if it has no separable odd cycles; (2) the minimum number q of vertex-disjoint odd cycles for which a maximum f-matching has fractional components, equals the maximum number s of vertex-disjoint odd cycles, separated by a maximum matching; (3) the difference between the cardinality of a maximum f-matching and that of a maximum matching in G is one half times s; (4) any maximum f-matching with fractional components for a minimum number s of vertex-disjoint odd cycles defines a maximum matching obtainable from it in s steps; and (5) if a maximum f-matching has fractional components for a set Q of odd cycles that is not minimum, there exists another maximum f-matching with fractional components for a minimum-cardinality set S of odd cycles, such that $S \subset Q$, $|Q \backslash S|$ is even, and the cycles in $Q \backslash S$ are pairwise connected by alternating paths.

INTEGER AND FRACTIONAL

ÉDGE-MATCHINGS

by

Egon Balas

1. Introduction

Let $G = (V,E)$ be an undirected graph with $|V| = m$ and $|E| = n$, let A be the vertex-edge incidence matrix of G, and e_p the p-vector of 1's. Consider the problem of finding a maximum-cardinality edge-matching in G

$$EM(G) \quad \max\{e_n x \mid Ax \le e_m, \ x \in \{0,1\}^n\}$$

and the associated linear program

$$LEM(G) \quad \max\{e_n x \mid Ax \le e_m, \ x \ge 0\}.$$

We will call $EM(G)$ the <u>matching problem</u>, and $LEM(G)$ the fractional matching, or shortly the <u>f-matching</u> problem. A feasible solution to $EM(G)$ will be called a <u>matching</u>, a feasible solution to $LEM(G)$, an <u>f-matching</u>. Clearly, every matching is an f-matching.

We wish to investigate the relationship between maximum matchings and maximum f-matchings in a graph G, i.e., between optimal solutions to $EM(G)$ and $LEM(G)$. For a problem P, we denote by $v(P)$ the value of (an optimal solution to) P. For vertex sets $S, T \subseteq V$, we denote by (S,T) the set of edges of G with one end in S and the other in T, and by $< S >$ the subgraph of G induced by S. Also, $\overline{S} = V \backslash S$.

An alternating path relative to a matching M is a path whose edges alternate between M and $\overline{M}$. An M-augmenting path is an alternating path whose end vertices are distinct and are not incident with M. A matching M in G is of maximum cardinality if and only if G contains no M-augmenting path (Berge [2]).

If G has no odd cycles (i.e., is bipartite), A is totally unimodular and every basic solution to LEM(G) is integer, i.e.,

(1) $v(LEM(G)) = v(EM(G))$.

If G is not bipartite, LEM(G) has fractional basic solutions, and the convex hull of feasible integer solutions, i.e., the matching polytope, is defined (together with the constraints of LEM(G)) by the inequalities

(2) $$\sum_{e\epsilon(S,S)} x_e \leq \frac{1}{2}(|S| - 1), \quad S \subseteq V, \quad |S| \geq 3 \text{ and odd}$$

(Edmonds [3]). Any basic feasible solution to LEM(G) has components equal to 0,1 or $\frac{1}{2}$ (Balinski [1]). Furthermore, if $x_e = \frac{1}{2}$ then e belongs to some odd cycle C, and $x_f = \frac{1}{2}$, $\forall$ $f\epsilon C$.

Let M be a maximum matching in G. If G is bipartite, M is a maximum f-matching; but the converse is false, as illustrated by the graph in figure 1, where the maximum matching shown in heavy lines is also a maximum f-matching, in spite of the presence of a 5-cycle.
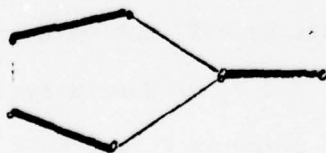

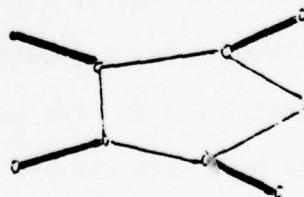Fig. 1   Fig. 2

It would seem that whether a maximum matching M is also a maximum f-matching, depends on whether M covers all vertices of every odd cycle; but this is not so. The matching in the graph of figure 2 leaves a vertex of an odd cycle exposed, yet it is a maximum f-matching. On the other hand, the

maximum matching M in the graph of figure 3a covers all vertices of the only odd cycle of the graph, yet it is not a maximum f-matching, as shown by the f-matching M' of figure 3b: $|M| = 3$, while the value of M' is $3\frac{1}{2}$.
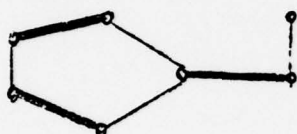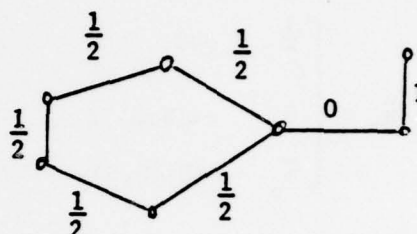
Fig. 3a   Fig. 3b

2. Separable Odd Cycles

The key concept for understanding the connection between maximum matchings and maximum f-matchings is the following.

Let M be a maximum matching and C an odd cycle in G, and let S be the vertex set of C. We will say that M separates C if

(3) $M \cap (S, \overline{S}) = \emptyset$.

An odd cycle which is separated by at least one maximum matching in G will be called separable. If C is a separable odd cycle in G, with vertex set S, then G has a maximum matching of the form $M = M_1 \cup M_2$, where M_1 and M_2 are maximum matchings in $< S >$ and in $< \overline{S} >$ respectively. If C is a nonseparable odd cycle with vertex set S, then every maximum matching contains at least k + 1 edges incident with S.

Theorem 1. G has a maximum f-matching that is integer if and only if G has no separable odd cycle.

-4-

Proof. Necessity. Let M be a maximum matching in G which separates an odd cycle C with vertex set S, and $|S| = 2k + 1$. Since M is maximal and $M \cap (S,\overline{S}) = \emptyset$, M contains k edges of (S,S). Let $\overline{x}$ be the incidence vector of M, and define $\hat{x}$ by

$$\hat{x}_e = \begin{cases} \frac{1}{2} & e \in C \\ 0 & e \in (S,S) \backslash C \\ \overline{x}_e & e \in E \backslash (S,S) \end{cases}$$

Clearly, $\hat{x}$ is a feasible solution to LEM(G), and

$$v(LEM(G)) \geq e_n \hat{x}$$

$$= |M| - k + \frac{1}{2}(k + 1) > v(EM(G)),$$

hence G has no maximum f-matching that is integer.

Sufficiency. Let $v(LEM(G)) > v(EM(G))$, and let $\hat{x}$ be a maximum f-matching in G having fractional components for a minimum number of odd cycles, say $C_1,...,C_q$. For $i = 1,...,q$, let S_i be the vertex set of C_i, with $|S_i| = 2k_i + 1$. Clearly, $S_i \cap S_j = \emptyset$, $\forall i,j \in \{1,...,q\}$. Further, let $S = \bigcup_{i=1}^{q} S_i$. The components $\hat{x}_e$ of $\hat{x}$ such that $\hat{x}_e = 1$ define a matching M' in $<\overline{S}>$ which is clearly maximum, or else $\hat{x}$ itself could not be a maximum f-matching.

For $i = 1,...,q$, let M_i be a set of k_i mutually nonadjacent edges of C_i, i.e., a maximum matching in $<S_i>$. Then the edge set

(4) $M = M' \cup M_1 \cup ... \cup M_q$

is a matching in G such that $M \cap (S_i,\overline{S}_i) = \emptyset$, $i = 1,...,q$.

Next we prove by contradiction that M is of maximum cardinality, which implies that the odd cycles $C_1,...,C_q$ are all separable and thus completes the proof of the theorem.

Suppose M is not maximum. Then there exists an M-augmenting path P in G. Moreover, we claim that the maximum matchings M_i in $< S_i >$, $i = 1,...,q$, which are not unique, can always be chosen such that P has no edge in common with any of the cycles C_i, $i = 1,...,q$. For suppose P contains an edge $(u,w) \epsilon C_i$ for some $i \epsilon \{1,...,q\}$. Since C_i has only one vertex exposed with respect to M, say t, at least one of the end vertices of the path P, say v, is not contained in C_i. Of the vertices u and w, let w be the one lying between u and v in P. Then the maximum matching M_i in $< S_i >$ can be replaced by $\widetilde{M}_i$, containing those k_i mutually nonadjacent edges of C_i that leave exposed w rather than t. This replaces the matching M in G by

$$\widetilde{M} = (M \backslash M_i) \cup \widetilde{M}_i ,$$

such that $|\widetilde{M}| = |M|$. The portion of P between t and w can then be removed, and the remaining portion, between w and v, is an $\widetilde{M}$-augmenting path $\widetilde{P}$ not containing (u,w). If $\widetilde{P}$ still has some edge in common with C_i, the above procedure can be repeated. After a finite number of applications of this procedure, we obtain a matching $\hat{M}$ in G such that $|\hat{M}| = |M|$,

$$\hat{M} = M' \cup \widetilde{M}_1 \cup ... \cup \widetilde{M}_q$$

(where each $\widetilde{M}_i$ is a maximum matching in $< S_i >$), and an $\hat{M}$-augmenting path $\hat{P}$, not containing any edge of any C_i, $i = 1,...,q$. This proves that w.l.o.g., the path P can be assumed to have no edge in common with any C_i, $i = 1,...,q$.

If we now reverse the assignment of edges in P with respect to M, we obtain a matching M* in G, such that $|M^*| = |M| + 1$. But from M* one can construct an f-matching $\bar{x}$ in G that has fractional components for fewer odd cycles than $\hat{x}$, contrary to our assumptions on $\hat{x}$.

To construct $\bar{x}$, first note that since P contains no edge of any C_i and is an alternating path, P contains no edge of $< S >$, and the only vertices of P contained in S are one, or possibly both, end vertices of P. The fact that P has at least one vertex in S follows from the maximality of the matching $M' = M \cap (\bar{S}, \bar{S})$ in $< \bar{S} >$.

Let v_i and v_j be the two end vertices of P. We consider two cases.

<u>Case 1</u>. $v_i \in S_i$ for some $i \in \{1, \ldots, q\}$, $v_j \notin S$. Let $e_1, \ldots, e_{2k+1}$ be the edges of C_i, and let e_1 and e_{2k+1} be incident with v_i. Define $\bar{x}$ by

$$\bar{x}_e = \begin{cases} 1 - \hat{x}_e & e \in P \\ 1 & e \in \{e_2, e_4, \ldots, e_{2k}\} \\ 0 & e \in \{e_1, e_3, \ldots, e_{2k+1}\} \\ \hat{x}_e & \text{otherwise} \end{cases}$$

<u>Case 2</u>. $v_i \in S_i$ and $v_j \in S_j$ for some $i, j \in \{1, \ldots, q\}$. Let e_r, $r = 1, \ldots, 2k_i + 1$, and d_s, $s = 1, \ldots, 2k_j + 1$, be the edges of C_i and C_j respectively, where e_1 and e_{2k_i+1} are incident with v_i, and d_1 and d_{2k_j+1} are incident with v_j. Then define $\bar{x}$ by

$$\bar{x}_e = \begin{cases} 1 - \hat{x}_e & e \in P \\ 1 & e \in \{e_2, e_4, \ldots, e_{2k_i}\} \cup \{d_2, d_4, \ldots, d_{2k_j}\} \\ 0 & e \in \{e_1, e_3, \ldots, e_{2k_i+1}\} \cup \{d_1, d_3, \ldots, d_{2k_j+1}\} \\ \hat{x}_e & \text{otherwise.} \end{cases}$$

In Case 1, $\bar{x}$ has fractional components for $q - 1$ odd cycles, and $e_n \bar{x} = e_n \hat{x} + \frac{1}{2}$, since the reversal of edge assignments in P adds 1 to, while the change of values for $e \in C_i$ subtracts $\frac{1}{2}$ from, the value of $e_n \hat{x}$. In case 2,

$\bar{x}$ has fractional components for $q - 2$ odd cycles, and $e_n\bar{x} = e_n\hat{x}$. In both cases the existence of $\bar{x}$ contradicts our assumption that $\hat{x}$ is a maximum f-matching which has fractional components for a minimum number (q) of odd cycles.

Thus M is a maximum cardinality matching in G, which separates the odd cycles $C_1,\ldots,C_q$. ||

The graphs in figures 1 and 2 both have maximum f-matchings that are integer, since in each case the unique maximum matching shown in heavy lines does not separate the unique odd cycle of the graph, hence the latter has no separable odd cycle. The graph of figure 3, on the other hand, has a maximum matching (other than the one shown in figure 3a) which separates the odd cycle: it contains two edges of the cycle, and a third edge not incident with any vertex of the cycle. Therefore there exists no maximum f-matching that is integer.

It is interesting to note that if LEM(G) has an integer optimum, it does not follow that the linear program dual to LEM(G), namely the fractional vertex covering problem LVC(G), also has an integer optimum. In other words, the absence of a separable odd cycle in G does not guarantee that the cardinality of a maximum edge matching is equal to that of a minimum vertex cover. For example the graph of figure 4 has no separable odd cycle and therefore it has a maximum f-matching of value 3 that is integer. On the other hand, the cardinality of a minimum vertex cover is 4, and the fractional vertex cover that assigns value $\frac{1}{2}$ to every vertex v is the only one with value 3.
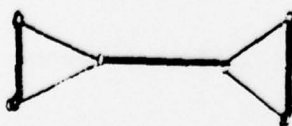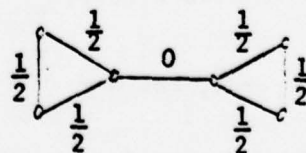

Fig. 4a

Fig. 4b

The proof of Theorem 1 is constructive in the sense that it gives a procedure for obtaining a maximum matching in an arbitrary graph G from a maximum f-matching in G that satisfies an additional requirement (having fractional components for a minimum number of odd cycles). It turns out that this procedure can be reversed, to obtain a maximum f-matching from a maximum matching that, again, satisfies an additional requirement: to separate a maximum number of disjoint odd cycles. (Two odd cycles are disjoint if they have no common vertex.) The next Corollary and its proof state this relationship more precisely.

For a maximum matching M, let $\sigma(G,M)$ be the maximum number of disjoint odd cycles separated by M. For a maximum f-matching x, let $\gamma(G,x)$ be the number of odd cycles for which x has fractional components. Finally, let $\mathcal{M}$ be the set of maximum matchings, and $\mathcal{X}$ the set of maximum f-matchings in G.

Corollary 1.1.

$$\max_{M \in \mathcal{M}} \sigma(G,M) = \min_{x \in \mathcal{X}} \gamma(G,x).$$

Proof. In the sufficiency part of the proof of Theorem 1, a maximum f-matching $\hat{x}$ such that

$$\gamma(G,\hat{x}) = q = \min_{x \in \mathcal{X}} \gamma(G,x)$$

was used to construct the matching defined by (4), which was shown to be maximum in G, and which separates q cycles of G. We claim that $q = \max_{M \in \mathcal{M}} \sigma(G,M)$. For suppose not, then there exists a maximum matching M* in G that separates $p \geq q + 1$ odd cycles $C_1,\ldots,C_p$. For $i = 1,\ldots,p$, let S_i be the vertex set of C_i, $|S_i| = 2k_i + 1$, let $S = \bigcup_{i=1}^{p} S_i$, and let $\bar{x}$ be the incidence vector of M*.

Then setting $\widetilde{x}_e = \frac{1}{2}$ for $e \in C_i$, $i = 1, \ldots, p$, $\widetilde{x}_e = \overline{x}_e$ otherwise, defines an f-matching $\widetilde{x}$ such that $e_n\widetilde{x} = |M^*| + \frac{1}{2} p$. On the other hand, $e\widetilde{x} = |M| + \frac{1}{2} q$, where M is the maximum matching defined by (4). Since $|M| = |M^*|$ and $p \geq q + 1$, this contradicts the assumption that $\hat{x}$ is a maximum f-matching in G. ‖

Next we turn to the relationship between the value of a maximum f-matching and that of a maximum matching in an arbitrary graph G. Let $\Delta(G)$ denote the difference of these two values, i.e.,

(5) $\Delta(G) = v(LEM(G)) - v(EM(G))$.

It is not hard to derive bounds on $\Delta(G)$. If C is an odd cycle with vertex set S, clearly $\Delta(< S >) = \frac{1}{2}$. Since in the absence of odd cycles in G, $\Delta(G) = 0$, if $\gamma(G)$ is the number of odd cycles in G, it is easy to see that

(6) $\Delta(G) \leq \frac{1}{2} \gamma(G)$.

This also yields a bound on $\Delta(G)$ independent of the number of odd cycles. Indeed, since a cycle has at least 3 edges, $\gamma(G) \leq \frac{1}{3} |E|$, i.e.,

(7) $\Delta(G) \leq \frac{1}{6} |E|$,

a bound which is attained for any graph G which is the union of triangles.

Further, since $\frac{1}{3} |E| \leq v(EM(G))$ obviously holds for any graph G, from (7) one also has

(8) $\frac{v(LEM(G))}{v(EM(G))} \leq \frac{3}{2}$.

The bounds (6), (7) and (8) are more or less trivial. The proof of Theorem 1 yields a considerably stronger bound, namely the actual value of $\Delta(G)$.

We define the separation number σ(G) of a graph G as the maximum number of disjoint odd cycles separated by any maximum matching in G, i.e.,

$$\sigma(G) = \max_{M \in \mathcal{M}} \sigma(G,M).$$

Corollary 1.2. $\Delta(G) = \frac{1}{2} \sigma(G)$.

Proof. Let $\hat{x}$ be a maximum f-matching in G such that $\gamma(G,\hat{x}) = \sigma(G)$. Then the matching M defined by (4) with respect to $\hat{x}$ is maximum, as shown in the proof of Theorem 1. Clearly, $|M| = e_n\hat{x} - \frac{1}{2} \sigma(G).\|$

The difference between the value of a maximum f-matching and that of a maximum matching can be considerably smaller than $\frac{1}{2} \gamma(G)$, the bound given by (6), since the separation number σ(G) of a graph can be much smaller than the number γ(G) of its odd cycles. Figure 5 shows a graph G with σ(G) = 0 and γ(G) = 2p, where p can be made arbitrarily large (in the figure, p = 5). The unique maximum matching, which is also a maximum f-matching in G, is shown in heavy lines.
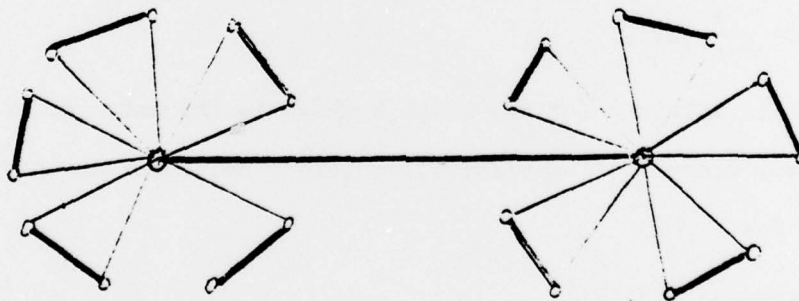

Fig. 5

For $S \subseteq V$, we denote by N(S) the set of vertices in $\overline{S}$ adjacent to some vertex in S. For $v \in V$, we denote by I(v) the set of edges incident with v in G.

<u>Corollary 1.3</u>. Every graph G has a maximum f-matching $\hat{x}$ such that

$$(9) \quad \hat{x}_e = \begin{cases} \frac{1}{2} & \forall\, e\epsilon C_i \,, & i = 1,\ldots,\sigma(G) \\[2mm] 0 & \forall\, e\epsilon(S_i,V)\backslash C_i \,, & i = 1,\ldots,\sigma(G) \\[2mm] 0 \text{ or } 1 & \forall\, e\epsilon(\overline{S},\overline{S}) \end{cases}$$

and

$$(10) \quad \sum_{e\epsilon I(v)} \hat{x}_e = 1 \,, \quad \forall\, v\epsilon N(S) \,,$$

where $C_1,\ldots,C_{\sigma(G)}$ is a maximum-cardinality set of disjoint separable odd cycles in G, S_i is the vertex set of C_i, $i = 1,\ldots,\sigma(G)$, and $S = S_1\cup\ldots\cup S_{\sigma(G)}$.

Conversely, every feasible solution $\hat{x}$ of LEM(G) that satisfies (9) is a maximum f-matching in G, and also satisfies (10).

<u>Proof</u>. From Corollary 1.1, G has a maximum f-matching $\hat{x}$ such that $\gamma(G,\hat{x}) = \sigma(G)$. Such an $\hat{x}$ clearly satisfies (9). Conversely, if $\hat{x}$ satisfies (9), then the matching M defined by (4) relative to $\hat{x}$ is maximum in G, hence $e_n\hat{x} = |M| + \frac{1}{2}\,\sigma(G) = v(EM(G)) + \frac{1}{2}\,\sigma(G)$ and thus $\hat{x}$ is maximal.

To show by contradiction that every maximum f-matching that satisfies (9) also satisfies (10), suppose (10) is violated for some $v\epsilon N(S)$. Let $u\epsilon S_i$ be the vertex of S adjacent to v, and let $e_1,\ldots,e_{2k_i+1}$ be the edges of C_i, with e_1 and e_{2k_i+1} incident with u.

Since (10) is violated for v, either $\hat{x}_e = 0$, $\forall\, e\epsilon I(v)$, or $\hat{x}_{e_*} = \frac{1}{2}$ for some $e_*\epsilon I(v)$ and $\hat{x}_e = 0$, $\forall\, e\epsilon I(v)\backslash\{e_*\}$. In the first case, the vector $\tilde{x}$ defined by

$$\tilde{x}_e = \begin{cases} 1 & e = (u,v) \\[2mm] 1 & e\epsilon\{e_2,e_4,\ldots,e_{2k_i}\} \\[2mm] 0 & e\epsilon\{e_1,e_3,\ldots,e_{2k_i+1}\} \\[2mm] \hat{x}_e & \text{otherwise} \end{cases}$$

is an f-matching with $e_n \tilde{x} = e_n \hat{x} + \frac{1}{2}$. In the second case, $\bar{x}$ defined by $\bar{x}_e = \tilde{x}_e$, $\forall$ $e \epsilon E \setminus \{e_*\}$, $\bar{x}_{e_*} = 0$, is an f-matching with $e_n \bar{x} = e_n \hat{x}$, but with fractional components for $\gamma(G,\hat{x}) - 1$ odd cycles. In both cases the outcome contradicts the assumption that $\hat{x}$ is a maximum f-matching with fractional components for a minimum number of odd cycles. ||

While every graph has a maximum f-matching $\hat{x}$ satisfying (9) (and hence (10)), it is not true that every (basic) maximum f-matching satisfies (9). In case of alternative optima there may be only one that satisfies the condition of Corollary 1.3. The graph G of figure 4, for instance, has the maximum f-matching shown in 4a, which satisfies (9), since G has no separable odd cycle; but it also has the one shown in 4b, which does not satisfy the condition of Corollary 1.3.

3. Generating Maximum Matchings from Maximum f-Matchings

Let $\hat{x}$ be a basic maximum f-matching, let

$$C(\hat{x}) = \{C_1, \ldots, C_q\}$$

be the set of odd cycles C_i such that $\hat{x}_e = \frac{1}{2}$, $\forall$ $e \epsilon C_i$, and for $i = 1, \ldots, q$, let S_i be the vertex set of C_i.

We will denote by $\mathcal{F}(\hat{x})$ the family of matchings associated with $\hat{x}$, introduced in the proof of Theorem 1, i.e., the set of all matchings of the form

$$M = M' \cup M_1 \cup \ldots \cup M_q,$$

where $M' = \{e \epsilon E | \hat{x}_e = 1\}$, and for $i = 1, \ldots, q$, M_i is a maximum matching in $< S_i >$.

We define two operations on the components of $\hat{x}$ associated with certain edge sets of G.

By <u>complementing</u> on $E' \subseteq E$, we mean replacing $\hat{x}_e$ by $\hat{x}'_e = 1 - \hat{x}_e$, $\forall$ $e \epsilon E'$.

By <u>alternate rounding</u> on C_i $\epsilon \, \mathcal{C}(\hat{x})$, $C_i = \{e_1, \ldots, e_{2k_i+1}\}$, we mean replacing $\hat{x}_e$, $e \epsilon C_i$, by $\hat{x}'_e = 0$, $e \epsilon \{e_1, e_3, \ldots, e_{2k_i+1}\}$, and $\hat{x}'_e = 1$, $e \epsilon \{e_2, \ldots, e_{2k_i}\}$.

From the results of the previous section it follows that one way of finding a maximum matching in a graph is to find a maximum f-matching $\hat{x}$ with a minimum-cardinality $\mathcal{C}(\hat{x})$ and then use alternative rounding on the odd cycles in $\mathcal{C}(\hat{x})$ to generate a matching $M \epsilon \mathcal{F}(\hat{x})$.

<u>Remark 1</u>. The matchings in $\mathcal{F}(\hat{x})$ are of maximum cardinality if and only if $\mathcal{C}(\hat{x})$ is of minimum cardinality.

<u>Proof</u>. From Corollaries 1.1 and 1.2.‖

Next we address the problem of finding a maximum f-matching $\hat{x}$ with a minimum-cardinality $\mathcal{C}(\hat{x})$.

<u>Lemma 1</u>. If $\hat{x}$ is a basic maximum f-matching and $q = |\mathcal{C}(\hat{x})| > \sigma(G)$, then

$$q \equiv \sigma(G) \pmod 2.$$

<u>Proof</u>. Let $M \epsilon \mathcal{F}(\hat{x})$. Then $e_n \hat{x} = |M| + \frac{1}{2} q$, and from Corollary 1.2, $e_n \hat{x} = v(EM(G)) + \frac{1}{2} \sigma(G)$. Since both $|M|$ and $v(EM(G))$ are integer, subtracting the second equation from the first one produces

$$\frac{1}{2}(q - \sigma(G)) \equiv 0 \pmod 1$$

and multiplication by 2 yields the congruence in the Lemma.‖

We define an <u>alternating path</u> relative to an f-matching $\hat{x}$ as a path whose edges e alternate between $\hat{x}_e = 0$ and $\hat{x}_e = 1$. We say that a path P connects two odd cycles C_1, C_2, if P connects a vertex of C_1 to a vertex of C_2. A vertex v of G is <u>exposed</u> relative to $\hat{x}$ if $\hat{x}_e = 0$ for all edges e incident with v.

Theorem 2. Let $\hat{x}$ be a basic maximum f-matching such that $q = |C(\hat{x})| > \sigma(G)$. Then G contains $\alpha = \frac{1}{2}(q - \sigma(G))$ vertex-disjoint alternating paths P_k relative to $\hat{x}$, each of which connects two odd cycles C_{i_k}, $C_{j_k} \in C(\hat{x})$, and the cycles C_{i_k}, C_{j_k}, $k = 1,\ldots,\alpha$, are all distinct. Furthermore, alternate rounding on the odd cycles C_{i_k}, C_{j_k}, and complementing on the paths P_k, $k = 1,\ldots,\alpha$, produces a maximum f-matching $\bar{x}$ with $|C(\bar{x})| = \sigma(G)$.

Proof. Let M^* be a maximum matching in G and $M \in \mathcal{F}(\hat{x})$. Since $|M| = e_n\hat{x} - \frac{1}{2}q$ and $|M^*| = e_n\hat{x} - \frac{1}{2}\sigma(G)$ (Corollary 1.2), $|M^*| - |M| = \frac{1}{2}(q - \sigma(G)) = \alpha$. It is therefore known (see Theorem 1 of [4]) that G contains α vertex-disjoint M-augmenting paths, $\pi_1,\ldots,\pi_\alpha$.

We show by contradiction that each π_k connects two cycles, C_{i_k}, $C_{j_k} \in C(\hat{x})$, $i_k \neq j_k$. Suppose this is false; then at least one end-vertex of π_k, say u, is in $\bar{S}$, where $S = \bigcup_{i=1}^{q} S_i$. Since $<\bar{S}>$ contains no M'-augmenting path (or else $\hat{x}$ would not be a maximum f-matching), π_k is incident with some vertex of S. Let v be the first vertex of S encountered when π_k is traversed starting from u, and let π_k' be the subpath of π_k connecting u to v. Clearly, π_k' is an alternating path relative to $\hat{x}$. Then using alternate rounding on C_i, the cycle whose vertex set S_i contains v, and complementing on π_k', produces an f-matching $\bar{x}$ such that $e_n\bar{x} = e_n\hat{x} + \frac{1}{2}$, contrary to the assumption that $\hat{x}$ is maximum.

This proves that both end-vertices of π_k belong to S. Further, since each S_i contains only one vertex exposed relative to M, the sets S_{i_k}, S_{j_k} containing the two end-vertices of π_k are distinct - hence so are the two cycles C_{i_k}, C_{j_k}; and for the same reason (that each S_i contains only one exposed vertex) the α pairs of odd cycles C_{i_k}, C_{j_k}, connected by the α paths π_k are all distinct.

Now using alternate rounding on the 2α odd cycles connected by the paths π_k reduces the value of the f-matching for every such cycle by $\frac{1}{2}$, i.e., for every pair of cycles C_{i_k}, C_{j_k} connected by a path π_k, by 1. On the other hand, using complementing on the paths π_k increases the value of the f-matching by 1 for every path π_k. Thus the alternate rounding and complementing produces an f-matching $\bar{x}$ of the same value as $\hat{x}$, hence maximum; and it reduces the number of odd cycles for which the f-matching has fractional components, by

$$2\alpha = q - \frac{1}{2}\,\sigma(G). \|$$

Two cycles C_i, C_j with vertex sets S_i, S_j will be called <u>adjacent</u>, if there exists $u \epsilon S_i$ and $v \epsilon S_j$ such that $(u,v) \epsilon E$. Since the edge (u,v) is an alternating path relative to $\hat{x}$, we have

<u>Corollary 2.1</u>. If two odd cycles C_i, $C_j \epsilon C(\hat{x})$ are adjacent, with $u \epsilon C_i$, $v \epsilon C_j$ and $(u,v) \epsilon E$, then alternate rounding on C_i and C_j and complementing on (u,v), produces a maximum f-matching $\bar{x}$ such that $|C(\bar{x})| = |C(\hat{x})| - 2$.

Also, from Theorem 2 we have

<u>Corollary 2.2</u>. If $|C(\hat{x})| > \sigma(G)$, there exists a maximum f-matching $\bar{x}$ such that $C(\bar{x}) \subset C(\hat{x})$ and $|C(\bar{x})| = \sigma(G)$.

Theorem 2 can be used to obtain from an arbitrary basic maximum f-matching, one with fractional components for a minimum number of odd cycles. The alternating paths relative to $\hat{x}$ can be found by a scanning and labeling procedure of the type used for finding augmenting paths relative to a matching. The complexity of such a scanning and labeling routine is $0(m^2)$ for each path found (where $m = |V|$), and since there are $\alpha = \frac{1}{2}(q - \sigma(G))$ paths to be found, obtaining from $\hat{x}$ a maximum f-matching $\bar{x}$ such that $|C(\bar{x})| = \sigma(G)$ requires at most $0(m^2 \cdot \alpha)$ steps.

The problem of finding a maximum f-matching in an arbitrary graph G is equivalent to the problem of finding a maximum matching in a bipartite graph $\hat{G} = (V_1 \cup V_2, E_1 \cup E_2)$, defined as follows. For every vertex $v_i \epsilon V$, $\hat{G}$ has a pair of vertices $v_{i1} \epsilon V_1$ and $v_{i2} \epsilon V_2$. For every edge $(v_i, v_j) \epsilon E$, G has a pair of edges $(v_{i1}, v_{j2}) \epsilon E_1$ and $(v_{i2}, v_{j1}) \epsilon E_2$. We say that the vertices v_{i1} and v_{i2}, as well as the edges (v_{i1}, v_{j2}) and (v_{i2}, v_{j1}), are copies of each other. If we associate the 0-1 vectors y^1 and y^2 with the edge sets E_1 and E_2 respectively, then a matching $(\hat{y}^1, \hat{y}^2)$ in $\hat{G}$ defines an f-matching $\hat{x}$ in G via the rule

$$\hat{x}_i = \begin{cases} 1 & \text{if } \hat{y}_i^1 = \hat{y}_i^2 = 1 \\ 0 & \text{if } \hat{y}_i^1 = \hat{y}_i^2 = 0 \\ \frac{1}{2} & \text{if } \hat{y}_i^1 = 0, \hat{y}_i^2 = 1, \text{ or } \hat{y}_i^1 = 1, \hat{y}_i^2 = 0. \end{cases}$$

Obviously, a maximum matching M in $\hat{G}$ then defines a maximum f-matching of cardinality $\frac{1}{2}|M|$ in G. Figure 6 shows a graph G with a maximum f-matching, and the associated graph $\hat{G}$ with the corresponding maximum matching.
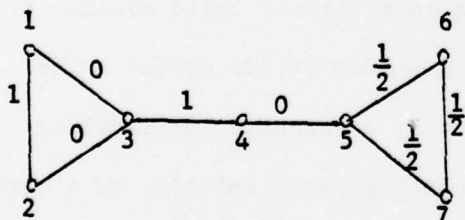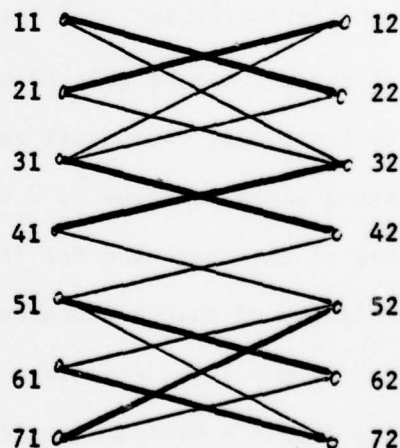

Fig. 6a

Fig. 6b

A bipartite graph of the above type was introduced by Edmonds and Pulleyblank (see Nemhauser and Trotter [6]) as an equivalent for the fractional vertex packing problem, but the equivalence obviously holds for the fractional matching problem LEM(G) as well.

Thus finding a maximum f-matching in an arbitrary graph G with n vertices is reduced to finding a maximum matching in a bipartite graph $\hat{G}$ with 2n vertices. Hopcroft and Karp [5] give an easy to implement $O(n^{5/2})$ algorithm for finding a maximum matching in a bipartite graph. An algorithm of the same complexity $(O(n^{5/2}))$, but much harder to implement, for maximum matching in an arbitrary graph, has been proposed by Even and Kariv [4]. Thus the worst case behavior of algorithms for maximum matching in a bipartite graph seems to be no better than that of their counterparts for arbitrary graphs. Nevertheless, the worst case is not the only, and perhaps not the most relevant one. The need for handling odd cycles in a matching algorithm for arbitrary graphs either by shrinking or by special labeling rules, make these algorithms considerably more difficult to implement than matching algorithms for bipartite graphs. Thus the idea of using the above described equivalence has some merit, even though it could not improve the worst-case bound. However, working out the details of an algorithm based on this approach is beyond the scope of our paper.

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
Technical Report No. 426		

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
Integer and Fractional Matchings	Technical Report February 1979
	6. PERFORMING ORG. REPORT NUMBER
	MSRR 426

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Egon Balas	N00014-75-C-0621 MPS73-08534 A02 NSF

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Graduate School of Industrial Administration Carnegie-Mellon University Pittsburgh, Pennsylvania 15213	NR 047-048

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
Personnel and Training Research Programs Office of Naval Research (Code 434) Arlington, Virginia 22217	February 1979
	13. NUMBER OF PAGES
	18

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

graphs, edge matching, integer and fractional matchings, separable odd cycles

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We examine the connections between maximum cardinality edge matchings in a graph and optimal solutions to the associated linear program, which we call maximum f-matchings (fractional matchings). We say that a maximum matching M separates an odd cycle with vertex set S, if M has no edge with exactly one end in S. An odd cycle is separable if it is separated by at least one maximum matching. We show that (1) a graph G has a maximum f-matching that is integer, if and only if it has no separable odd cycles; (2) the minimum ⟶

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number q of vertex-disjoint odd cycles for which a maximum f-matching has fractional components, equals the maximum number s of vertex-disjoint odd cycles, separated by a maximum matching; (3) the difference between the cardinality of a maximum f-matching and that of a maximum matching in G is one half times s; (4) any maximum f-matching with fractional components for a minimum number s of vertex-disjoint odd cycles defines a maximum matching obtainable from it in s steps; and (5) if a maximum f-matching has fractional components for a set Q of odd cycles that is not minimum, there exists another maximum f-matching with fractional components for a minimum-cardinality set S of odd cycles, such that S⊆Q, |Q\S| is even, and the cycles in Q\S are pairwise connected by alternating paths.

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